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DETECTION, ESTIMATION, AND CONTROL ON GROUP MANIFOLDS.(U)
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Final Scientific Report

on

Detection, Estimation, and Control

on Group Manifolds

TO: Air Force Office of Scientific Research
Air Force Systems Command
USAF

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SUBJECT: Contract No. F49620-79-C-0186
Detection, Estimation, and Control
on Group Manifolds

covering the period:
July 1, 1979 - July 31, 1980

I. General

This final report covers work carried out by the principal investigator and a graduate research assistant at the Department of Mathematics during the 13 months period from July 1, 1979 to July 31, 1980 under Contract F49620-79-C-0186.

The progress has resulted in three technical papers listed in Section II, which will be submitted for publication as soon as completed.

During the report period, the following people contributed to the project: Professor James T. Lo and Mr. Sze-Kui Ng.

II. Publications

(1) Transition Probabilities of Homogeneous Markov Processes on Compact Lie Groups, to appear.

(2) Projecting Homogeneous Markov Processes onto Compact Homogeneous Spaces, to appear.

(3) Approximation of Brownian Motion Densities on the Three Dimensional Rotation Group, to appear.

III. Summary of Progress

An optimal estimation scheme for continuous-time rotational processes with one degree of freedom was obtained in [1]. The estimation problem for continuous-time rotational processes with three degrees of freedom turns out to be rather difficult. Many attempts were made by the P.I. in the past few years on the problem. While these attempts produced minimal results, many crucial issues have been placed in perspective. Because of the theoretical and practical importance of the problem, a concentrated effort was finally made during the report period to study these issues.

(1) Transition Probabilities of Homogeneous Markov Processes on Compact Lie Groups:

A signal process on SO(3)(the 3-dimensional rotation group) can be constructed by injecting a 3-dimensional Brownian motion into SO(3) [2]. *It is known that such a signal process is governed by* a bilinear stochastic differential equation. But how do we compute the transition density for this process? In the SO(2) case, such a transition density has an explicit expression called the folded normal density. It is also called a theta function. Then what is a general folded normal density or theta function? This question is answered in a much more general context as follows:

Let $\{T_g^{(n)}\}$ be a sequence of finite dimensional irreducible unitary representations of a k-dimensional compact Lie group G where $T_g^{(1)} = 1$. The infinitesimal operators $A_i^{(n)}$ of $T_g^{(n)}$ is defined as

$$A_i^{(n)} = \lim_{t \downarrow 0} \frac{1}{t}\left[T_{g_i(t)}^{(n)} - I_n\right], \qquad i = 1,\ldots, k$$

where $\{g_i\}$ are k one-parameter subgroups of which the tangent vectors at the group identity e are orthogonal. There is a neighborhood of e in which a parameterization $x_1(g),\ldots, x_k(g)$ satisfies

$$T_g^{(n)} = \exp\left(\sum_{i=1}^{k} x_i(g) A_i^{(n)}\right), \qquad \forall g \in G.$$

These local coordinates $\{x_i(g)\}$ can be extended continuously to the entire G such that they do not vanish simultaneously except at e. Let X_i denote the left invariant vector fields corresponding to g_i. Hung [3] proved that for a smooth function f and the transition probability P(t, g, E) of a homogeneous Markov process, the function $u(t, g) = \int_G f(\sigma)P(t, g, d\sigma)$ satisfies the Kolmogrov backward equation, with P(0, e, E) concentrates at e.

$$\frac{\partial u(t, g)}{\partial t} = \sum_{i,j=1}^{k} a_{ij}X_iX_ju(t, g) + \sum_{i=1}^{k} a_iX_iu(t, g)$$

$$\int_{G\setminus\{e\}}\left[u(t, gh) - u(t, g) - \sum_{i=1}^{k} x_i(h)X_iu(t, g)\right]dF(h),$$

for a real vector $[a_1,\ldots, a_k]$, a non-negative definite matrix $[a_{ij}]_{i,j=1}^{k}$, and a positive measure F on $G\setminus\{e\}$ such that

$$\int_{G\setminus\{e\}}\sum_{i=1}^{k} x_i^2(g)dF(g) < \infty.$$

Our result is that the transition probability P(t, g, E) can be calculated by using the following fact:

Let $$C_n(t)=\exp t\left\{\sum_{i,j=1}^{k} a_{ij}A_i^{(n)}A_j^{(n)} + \sum_{i=1}^{k} a_iA_i^{(n)}\right.$$

$$\left. + \int_{G\setminus\{e\}}\left[T_g^{(n)} - I_n - \sum_{i=1}^{k} x_i(g)A_i^{(n)}\right]dF(g)\right\}$$

and ρ_n = (dim. of $T_g^{(n)}$)/(the volume of G).

Under certain conditions, there exists a sequence of functions

$$f_r(t, g, \sigma) = \sum_{n=0}^{N(r)} \rho_n\lambda_n^r \operatorname{tr}\left[T_g^{(n)*}T_\sigma^{(n)}C_n^*(t)\right]$$

such that for $h\in C(G)$,

$$\lim_{r\to\infty}\int h(\sigma)f_r(t,\ g,\ \sigma)d\sigma = \int h(\sigma)P(t,\ g,\ d\sigma)$$

where $|\lambda_n^r| \leq 1$, $\lambda_0^r = 1$, $\lim_{r\to\infty} \lambda_n^r = 1$, $n = 0,1,\ldots$ and $\lim_{r\to\infty} N(r) = \infty$.

For example, let us consider $G = SO(3)$ and the Brownian motion governed by the bilinear Ito equation,

$$dX(t) = \begin{bmatrix} -dt & -dw_3 & dw_2 \\ dw_3 & -dt & -dw_1 \\ -dw_2 & dw_1 & -dt \end{bmatrix} X(t),\ X(0) = I$$

where $[w_1,\ w_2,\ w_3]$ is the standard Brownian motion in R^3. The transition density $p(t,\ g,\ \sigma)$ of the process satisfies

$$\frac{\partial p}{\partial t}(t,\ g,\ \sigma)$$
$$= \frac{1}{\sin\beta}\ \frac{\partial}{\partial\beta}\left\{\sin\beta\frac{\partial p}{\partial\beta}(t,\ g,\ \sigma)\right\} + \frac{1}{\sin^2\beta}\left\{\frac{\partial^2 p}{\partial\lambda^2}(t,\ g,\ \sigma)\right.$$
$$\left. - 2\cos\beta\frac{\partial^2 p}{\partial\alpha\partial\gamma}(t,\ g,\ \sigma) + \frac{\partial^2 p}{\partial\alpha^2}(t,\ g,\ \sigma)\right\}$$
$$p(0,\ e,\ \sigma) = \delta(\sigma)$$

where $g = (\alpha',\ \beta',\ \gamma')$ and $\sigma = (\alpha,\ \beta,\ \gamma)$. It can be explicitly written as

$$p(t,\ g,\ \sigma) = \sum_{j=0}^{\infty} \frac{2n+1}{8\pi^2}\ e^{-j(j-1)t} \sum_{m,n=-j}^{j} e^{im\alpha'}\overline{P^j_{mn}}(\cos\beta')e^{in\gamma'}e^{-im\alpha}$$
$$\cdot P^j_{mn}(\cos\ \beta)e^{-in\gamma}$$

where $e^{-im\alpha}P^j_{mn}(\cos\ \beta)e^{-in\gamma}$ is the generalized spherical functions of order j.

(2) Projecting Homogeneous Markov Processes onto Compact Homogeneous Spaces:

The 2-sphere S^2, the surface of a ball, is a natural state space for the directional processes. It is isomorphic to the quotient space $SO(3)/SO(2)$. A signal process on S^2 can be induced by projecting a signal process on $SO(3)$ onto S^2. If the signal process on $SO(3)$ is Brownian, then what is the transition density of the induced Brownian motion on S^2? This question is also answered in a general context as follows:

Let H be a closed subgroup of the Lie group G. Corresponding to a projection $\Pi: g \to Hg$ from G onto the homogeneous space $H\backslash G$, there exists a C^∞ mapping s such that $\Pi \circ s = 1$. For each s, an element $g \in G$ is expressed uniquely as $s_1(x)h$ for some $x \in H\backslash G$ and $h \in H$. A homogeneous process on $H\backslash G$ can be induced by projecting through Π a homogeneous process on G.

The transition probability $P(t, x, B)$ of the resulting process on $H\backslash G$ can be calculated by using the following fact:

Under certain conditions, there exists a sequence of functions

$$f_r(t, x, y) = \sum_{n=0}^{N(r)} \rho_n \lambda_n^r \, \mathrm{tr}\left[T_{s(x)}^{(n)*} \int_H T_g^{(n)} dg T_{s(y)}^{(n)} C_n^*(t)\right]$$

such that for $h \in C(H\backslash G)$,

$$\lim_{r \to \infty} \int_{H\backslash G} h(y) f_r(t, x, y) d\mu(y) = \int_{H\backslash G} h(y) P(t, x, dy)$$

where μ is an invariant measure on $H\backslash G$ and the other symbols are defined as in (1).

For example, the transition density of the homogeneous process that is the projection of the example in (1) satisfies

$$\frac{\partial p}{\partial t}(t, x, y) = \frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\left\{\sin\beta\frac{\partial p}{\partial\beta}(t, x, y)\right\} + \frac{1}{\sin^2\beta}\frac{\partial^2 p}{\partial\gamma^2}(t, x, y)$$

$$p(0, 0, y) = \delta(y)$$

where $x = (\alpha', \beta')$ and $y = (\alpha, \beta)$, the spherical coordinates. It can be explicitly written as

$$p(t, x, y) = \sum_{k=1}^{\infty}\frac{2k+1}{4\pi}e^{-k(k+1)t}\sum_{n=-k}^{k}\overline{Y}_k^n(\tfrac{\pi}{2} - \gamma', \beta')Y_k^n(\tfrac{\pi}{2} - \gamma, \beta)$$

where $Y_k^n(\frac{\pi}{2} - \gamma, \beta)$, $-k \leq n \leq k$ are the unnormalized spherical functions of the k-th order.

(3) Approximation of Brownian Motion Densities on the Three Dimensional Rotation Group:

The transition densities obtained above are very complicated functions, as illustrated by the explicit expressions for the transition densities of the simplest possible Brownian motions on SO(3) and S^2. In order to use them for estimation, a natural question is whether they can be approximated closely by simpler closed-form functions.

Let us restrict our attention to the density obtained in the example in (1):

$$p_\sigma(\alpha, \beta, \gamma) = p(t, e, (\alpha, \beta, \gamma)) = \sum_{\ell=0}^{\infty}\frac{2\ell+1}{8\pi}e^{-\sigma\ell(\ell+1)}\sum_{m=-\ell}^{\ell}e^{-im(\alpha+\gamma)}P_{mm}^{\ell}(\cos\beta).$$

It can be approximated closely by

$$p_k(\alpha, \beta, \gamma) = \frac{1}{2\pi C_0(k)C_1(k)} e^{k \cos \beta + k \cos (\alpha + \gamma)}$$

$$C_0(k) = \int_0^{2\pi} e^{k \cos \alpha} d\alpha$$

$$C_1(k) = \int_0^{\pi} e^{k \cos \beta} d\beta$$

where k is determined by $\coth k - \frac{1}{k} = e^{-2\sigma}$. Let the Fourier series expansion of p_k be

$$p_k(\alpha, \beta, \gamma) = \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{8\pi^2} \operatorname{tr}\left(T_g^{(\ell)} C_\ell^*(k)\right)$$

where $C_\ell(k) = \left(C^\ell_{nm}(k)\right)$ is a $2\ell + 1$ dimensional coefficient matrix. It is proven that

(a) For each m, ℓ such that $-\ell \leq m \leq \ell$,

$$C^\ell_{mm}(k) = e^{-\sigma\ell(\ell + 1)} + 0\left(\frac{1}{k^2}\right)$$

$$C^\ell_{mm}(k) = 0, \qquad \text{for } n \neq m$$

(b) Given any $\varepsilon > 0$ and positive integer N, there exists a number $K(\varepsilon, N)$ such that, for

$$k \geq K(\varepsilon, N), \quad \ell^2 \leq Nk, \text{ and } -\ell \leq m \leq \ell ,$$

$$\left| C^\ell_{mm}(k) - e^{-\sigma\ell(\ell + 1)} \right| \leq \varepsilon.$$

It is noted that (a) is analogous to those for spheres and circles obtained in [4] and [5]. (b) is new and stronger than (a). (b) for spheres and circles is also obtained in our work.

References

[1] J. T. Lo and A. S. Willsky, "Estimation for Rotational Processes with One Degree of Freedom I: Introduction and Continuous-Time Processes," IEEE Transactions on Automatic Control, AC-20-1, February 1975.

[2] J. T. Lo, "Representation of Continuous Curves on the 3-dimensional Rotation Group," Mathematical System Theory, 8-4, pp. 368-375, 1975.

[3] G. A. Hunt, "Semi-groups of Measures on Lie Groups," Trans. Amer. Math. Soc., 81, pp. 264-293, 1956.

[4] P. H. Roberts and H. D. Ursell, "Random Walk on a Sphere and on a Riemannian Manifold," Philos. Trans. A, 252, pp. 317-56, 1960.

[5] M. A. Stevens, "Random Walk on a Circle," Biometrika 50, 3 and 4, pp. 385-390, 1963.

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